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Halfconvex functions and equality of Daróczy–Páles conjugate means

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Abstract. Using some properties of halfconvex and halfconcave functions, we solve the equality problem for conjugate means introduced by Daróczy and Páles.

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1. Introduction

For a mean $K : I^2 \to I$, a continuous strictly monotonic function φ defined in an interval $I \subset \mathbb{R}$, and the numbers $p, q \in [0, 1]$, Daróczy and Páles [3] defined a K-*conjugate mean* $K_\varphi^{(p,q)}$ by the formula

$$K_\varphi^{(p,q)}(x, y) := \varphi^{-1}(p\varphi(x) + q\varphi(y) + (1 - p - q)\varphi(K(x, y))), \quad x, y \in I.$$

In the case $p = q = 1$ such a mean was considered in [6].

In [2] the authors solve the equality problem of a conjugate mean and a weighted quasi-arithmetic mean. It is interesting that this equality problem contains the invariance equation, considered among others in [4,6], as a special case.

In the present paper we consider the question of equality of conjugate means, that is, we deal with the equation $L_\psi^{(r,s)} = K_\varphi^{(p,q)}$. This problem easily reduces to the equality $L_{id}^{(r,s)} = K_\gamma^{(p,q)}$ where $\gamma := \varphi \circ \psi^{-1}$ (Sect. 3). In Sect. 4 we show that this leads to inequalities of the form

$$\gamma((1 - s)x + sy) \le p\gamma(x) + (1 - p)\gamma(y), \quad x, y \in I, \ x \le y;$$

and

$$\gamma((1 - s)x + sy) \ge p\gamma(x) + (1 - p)\gamma(y), \quad x, y \in I, \ x \le y$$

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(Lemmas 4.1–4.4). These weaker forms of (s,p)-convexity ((s,p)-concavity) of a function appeared for the first time in a recent paper of Daróczy [1], where the function satisfying the first of these inequalities is said to be (s,p)-*halfconvex*. Some auxiliary results on the properties of (s,p)-halfconvex and (s,p)-halfconcave functions presented in Sect. 5 allow us to prove the main result which reads as follows: *if $K, L : I^2 \to I$ are means in the interval I, $p, q, r, s \in [0,1]$ are fixed numbers such that $p+q \neq 1 \neq r+s$, and $\varphi, \psi : I \to R$ are continuous strictly monotonic functions, then $L_\psi^{(r,s)} = K_\varphi^{(p,q)}$ if, and only if, $\psi \sim \varphi$ and, for all $x, y \in I$,*

$$L(x,y) = \varphi^{-1}\left(\frac{(p-r)\varphi(x) + (q-s)\varphi(y) + (1-p-q)\varphi(K(x,y))}{1-r-s}\right).$$

2. Preliminaries

For an interval $I \subset \mathbb{R}$, by $\mathcal{CM}(I)$ denote the family of all continuous and strictly monotonic functions $\varphi : I \to \mathbb{R}$, and by $\mathcal{CM}_+(I)$ the family of all continuous and strictly increasing functions $\varphi : I \to \mathbb{R}$. The functions $\varphi, \psi \in \mathcal{CM}(I)$ are called equivalent, in writing $\psi \sim \varphi$, if $\psi = a\varphi + b$ for some $a, b \in \mathbb{R}, a \neq 0$.

Recall that a function $M : I^2 \to I$ is called a mean if

$$\min(x,y) \leq M(x,y) \leq \max(x,y), \quad x, y \in I.$$

A mean M is called strict if for all $x, y \in I, x \neq y$, these inequalities are sharp; symmetric if $M(x,y) = M(y,x)$.

Let $I \subset \mathbb{R}$ be an interval, $K : I^2 \to I$ a mean in I, $p, q \in [0,1]$ fixed numbers and $\varphi \in \mathcal{CM}(I)$. Then (cf. [3]) the function $K_\varphi^{(p,q)} : I^2 \to \mathbb{R}$ defined by

$$K_\varphi^{(p,q)}(x,y) := \varphi^{-1}(p\varphi(x) + q\varphi(y) + (1-p-q)\varphi(K(x,y))), \quad x, y \in I,$$

is a mean and it is called the K-conjugate mean of the quasi-generator φ, and the weights p, q. Note that

$$K_\varphi^{(0,0)} = K \quad \text{and} \quad K_\varphi^{(p,1-p)} = A_p^{[\varphi]},$$

where

$$A_p^{[\varphi]}(x,y) := \varphi^{-1}(p\varphi(x) + (1-p)\varphi(y)), \quad x, y \in I,$$

is the quasi-arithmetic mean of the generator φ and the weight p (cf. [2]).

Thus the family of means $\{K_\varphi^{(p,q)} : p, q \in [0,1]\}$ contains the mean K and all the weighted quasi-arithmetic means $A_p^{[\varphi]}$. Moreover, from the equality

$$K_\varphi^{(1,1)}(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y) - \varphi(K(x,y))), \quad x, y \in I,$$

we get

$$\varphi(K(x,y)) + \varphi(K_\varphi^{(1,1)}(x,y)) = \varphi(x) + \varphi(y), \quad x, y \in I,$$

whence, setting $A^{[\varphi]} := A^{[\varphi]}_{1/2}$, we obtain

$$A^{[\varphi]} \circ (K, K_{\varphi}^{(1,1)}) = A^{[\varphi]},$$

that is, the quasi-arithmetic mean $A^{[\varphi]}$ is invariant with respect to the mean-type mapping $(K, K_{\varphi}^{(1,1)}) : I^2 \to I^2$ (cf. [6], Remark 2). This fact is important in applications.

Remark 2.1. If $\varphi, \psi \in \mathcal{CM}(I)$ and $\varphi \sim \psi$, then $K_{\psi}^{(p,q)} = K_{\varphi}^{(p,q)}$. Therefore we can always assume that the generator φ of the mean $K_{\varphi}^{(p,q)}$ is strictly increasing, i.e. that $\varphi \in \mathcal{CM}_{+}(I)$.

3. Problem and remarks

If $L : I^2 \to I$ is another mean in I, the numbers $r, s \in [0,1]$ are fixed, and $\psi \in \mathcal{CM}(I)$, then we can ask when

$$L_{\psi}^{(r,s)} = K_{\varphi}^{(p,q)},$$

that is, when two conjugate means of different generators and weights can be equal?

The following is easy to verify.

Remark 3.1. Assume that $I \subset \mathbb{R}$ is an interval, $K, L : I^2 \to I$ are means, $p, q, r, s \in [0,1]$ are fixed numbers, and $\varphi, \psi \in \mathcal{CM}(I)$. Then

$$L_{\psi}^{(r,s)} = K_{\varphi}^{(p,q)}$$

if, and only if,

$$\mathcal{L}_{id}^{(r,s)} = \mathcal{K}_{\gamma}^{(p,q)},$$

where $\mathcal{L}, \mathcal{K} : \psi(I) \times \psi(I) \to \psi(I)$ are means defined by

$$\mathcal{K}(u,v) := \psi(K(\psi^{-1}(u), \psi^{-1}(v))), \quad \mathcal{L}(u,v) := \psi(L(\psi^{-1}(u), \psi^{-1}(v)))$$

for all $u, v \in \psi(I)$, and

$$\gamma := \varphi \circ \psi^{-1};$$

that is

$$\mathcal{K}_{\gamma}^{(p,q)}(u,v) = \gamma^{-1}(p\gamma(u) + q\gamma(v) + (1-p-q)\gamma(\mathcal{K}(u,v))), \quad u, v \in \psi(I),$$

and

$$\mathcal{L}_{id}^{(r,s)} = ru + sv + (1-r-s)\mathcal{L}(u,v), \quad u, v \in \psi(I).$$

This remark allows us to reduce the problem of equality of conjugate means to the functional equation $L_{id}^{(r,s)} = K_{\gamma}^{(p,q)}$, where *id* denotes the identity function and γ is continuous and strictly monotonic.

Remark 3.2. Let $I \subset \mathbb{R}$ be an interval and $p, q, r, s \in [0, 1]$. Suppose that $\gamma \in \mathcal{CM}(I)$ and the functions $K, L : I^2 \to I$ are means in I.

(i) If $p + q = 1$ and $r + s = 1$, then $L_{id}^{(r,s)} = K_\gamma^{(p,q)}$ iff $\gamma \sim id$ and $r = p, s = q$. Moreover, in this case both means K and L can be arbitrary.

(ii) If $p + q = 1$ and $r + s \neq 1$, then $L_{id}^{(r,s)} = K_\gamma^{(p,q)}$ iff

$$L(x, y) = \frac{\gamma^{-1}(p\gamma(x) + (1 - p)\gamma(y)) - rx - sy}{1 - r - s}, \quad x, y \in I.$$

Moreover, in this case the mean K can be arbitrary.

(iii) If $p + q \neq 1$ and $r + s = 1$, then $L_{id}^{(r,s)} = K_\gamma^{(p,q)}$ iff

$$K(x, y) = \gamma^{-1}\left(\frac{\gamma(rx + (1 - r)y) - p\gamma(x) - q\gamma(y)}{1 - p - q}\right), \quad x, y \in I.$$

Moreover, in this case the mean L can be arbitrary.

Proof. (i) Since $q = 1 - p, s = 1 - r$, by the definition of conjugate means, the equality $L_{id}^{(r,s)} = K_\gamma^{(p,q)}$ reduces to

$$rx + (1 - r)y = \gamma^{-1}(p\gamma(u) + (1 - q)\gamma(v)), \quad x, y \in I,$$

which holds true iff $\gamma \sim id$ and $r = p, s = q$ (cf. [5]).

The cases (ii) and (iii) follow directly from the definitions of the means $L_{id}^{(r,s)}$ and $K_\gamma^{(p,q)}$. $\square$

4. Equality of conjugate means implies halfconvexity and halfconcavity of quasi-generator

We begin this section with the following

Lemma 4.1. *Let $I \subset \mathbb{R}$ be an interval and $p, q, r, s \in [0, 1]$ be such that $p + q < 1$ and $r + s < 1$. Suppose that $\gamma \in \mathcal{CM}_+(I)$, the functions $K, L : I^2 \to I$ are means in I, and*

$$L_{id}^{(r,s)} = K_\gamma^{(p,q)}. \tag{1}$$

Under these assumptions, for all $x, y \in I$,

(i) *if $x \leq y$ then*

$$\gamma((1 - s)x + sy) \leq p\gamma(x) + (1 - p)\gamma(y),$$

$$\gamma(rx + (1 - r)y) \geq (1 - q)\gamma(x) + q\gamma(y);$$

(ii) *if $x > y$ then*

$$\gamma(rx + (1 - r)y) \leq (1 - q)\gamma(x) + q\gamma(y),$$

$$\gamma((1 - s)x + sy) \geq p\gamma(x) + (1 - p)\gamma(y).$$

Proof. If (1) holds, that is, if

$$rx + sy + (1 - r - s)L(x, y) = \gamma^{-1}(p\gamma(x) + q\gamma(y) + (1 - p - q)\gamma(K(x, y)))$$

for all $x, y \in I$, then

$$L(x, y) = \frac{\gamma^{-1}(p\gamma(x) + q\gamma(y) + (1 - p - q)\gamma(K(x, y))) - rx - sy}{1 - r - s}, \quad x, y \in I,$$

whence, for all $x, y \in I$,

$$\begin{aligned} \min(x, y) &\leq \frac{\gamma^{-1}(p\gamma(x) + q\gamma(y) + (1 - p - q)\gamma(K(x, y))) - rx - sy}{1 - r - s} \\ &\leq \max(x, y). \end{aligned} \tag{2}$$

Since $r + s < 1$ and $p + q < 1$, we get

$$\begin{aligned} &\gamma^{-1}\left(\frac{\gamma((1 - r - s)\min(x, y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1 - p - q}\right) \\ &\quad \leq K(x, y) \leq \gamma^{-1}\left(\frac{\gamma((1 - r - s)\max(x, y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1 - p - q}\right) \end{aligned}$$

for all $x, y \in I$. Since K is a mean, it follows that

$$\gamma^{-1}\left(\frac{\gamma((1 - r - s)\min(x, y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1 - p - q}\right) \leq \max(x, y)$$

for all $x, y \in I$, and

$$\gamma^{-1}\left(\frac{\gamma((1 - r - s)\max(x, y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1 - p - q}\right) \geq \min(x, y),$$

for all $x, y \in I$, whence, as γ is strictly increasing,

$$\gamma((1 - r - s)\min(x, y) + rx + sy) \leq (1 - p - q)\gamma(\max(x, y)) + p\gamma(x) + q\gamma(y)$$

for all $x, y \in I$, and

$$\gamma((1 - r - s)\max(x, y) + rx + sy) \geq (1 - p - q)\gamma(\min(x, y)) + p\gamma(x) + q\gamma(y)$$

for all $x, y \in I$. Now, if $x < y$, from these inequalities, we get

$$\begin{aligned} \gamma((1 - s)x + sy) &\leq p\gamma(x) + (1 - p)\gamma(y), \\ \gamma(rx + (1 - r)y) &\geq (1 - q)\gamma(x) + q\gamma(y); \end{aligned}$$

and, if $x > y$, we get

$$\begin{aligned} \gamma(rx + (1 - r)y) &\leq (1 - q)\gamma(x) + q\gamma(y), \\ \gamma((1 - s)x + sy) &\geq p\gamma(x) + (1 - p)\gamma(y). \end{aligned}$$

□

In the sequel of this section we consider three remaining cases.

Lemma 4.2. *Let $I \subset \mathbb{R}$ be an interval and $p, q, r, s \in [0, 1]$ be such that $p+q > 1$ and $r+s > 1$. Suppose that $\gamma \in \mathcal{CM}(I)$, the functions $K, L : I^2 \to I$ are means in I, and*

$$L_{id}^{(r,s)} = K_{\gamma}^{(p,q)}.$$

Under these assumptions, for all $x, y \in I$,

(i) *if* $x \leq y$ *then*

$$\gamma((1-s)x + sy) \geq p\gamma(x) + (1-p)\gamma(y),$$
$$\gamma(rx + (1-r)y) \leq (1-q)\gamma(x) + q\gamma(y);$$

(ii) *if* $x > y$ *then*

$$\gamma(rx + (1-r)y) \geq (1-q)\gamma(x) + q\gamma(y),$$
$$\gamma((1-s)x + sy) \leq p\gamma(x) + (1-p)\gamma(y).$$

Proof. Similarly as in the proof of Lemma 4.1 we can assume that γ is an increasing function. Calculating $L(x, y)$ from the equality $L_{id}^{(r,s)} = K_{\gamma}^{(p,q)}$ and taking into account that $\min(x, y) \leq L(x, y) \leq \max(x, y)$, we obtain inequalities (2) for all $x, y \in I$. Since $r + s > 1$ and $p + q > 1$, we get

$$\gamma^{-1}\left(\frac{\gamma((1-r-s)\min(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right)$$
$$\leq K(x,y) \leq \gamma^{-1}\left(\frac{\gamma((1-r-s)\max(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right)$$

for all $x, y \in I$. Since K is a mean, it follows that

$$\gamma^{-1}\left(\frac{\gamma((1-r-s)\min(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right) \leq \max(x,y)$$

for all $x, y \in I$, and

$$\gamma^{-1}\left(\frac{\gamma((1-r-s)\max(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right) \geq \min(x,y),$$

for all $x, y \in I$. Hence, as $1 - p - q < 0$, we get

$$\gamma((1-r-s)\min(x,y) + rx + sy) \geq (1-p-q)\gamma(\max(x,y)) + p\gamma(x) + q\gamma(y),$$

$x, y \in I$, and

$$\gamma((1-r-s)\max(x,y) + rx + sy) \leq (1-p-q)\gamma(\min(x,y)) + p\gamma(x) + q\gamma(y),$$

$x, y \in I$. Now, if $x < y$, from these inequalities, we get

$$\gamma((1-s)x + sy) \geq p\gamma(x) + (1-p)\gamma(y),$$
$$\gamma(rx + (1-r)y) \leq (1-q)\gamma(x) + q\gamma(y);$$

and, if $x > y$, we get

$$\gamma(rx + (1-r)y) \geq (1-q)\gamma(x) + q\gamma(y),$$
$$\gamma((1-s)x + sy) \leq p\gamma(x) + (1-p)\gamma(y).$$

□

Lemma 4.3. *Let $I \subset \mathbb{R}$ be an interval and $p, q, r, s \in [0, 1]$ be such that $p+q < 1$ and $r + s > 1$. Suppose that $\gamma \in \mathcal{CM}_+(I)$, the functions $K, L : I^2 \to I$ are means in I, and*

$$L_{id}^{(r,s)} = K_{\gamma}^{(p,q)}.$$

Under these assumptions, for all $x, y \in I$,

(i) *if $x \leq y$ then*

$$\gamma(rx + (1-r)y) \leq p\gamma(x) + (1-p)\gamma(y),$$
$$\gamma((1-s)x + sy) \geq (1-q)\gamma(x) + q\gamma(y);$$

(ii) *if $x > y$ then*

$$\gamma((1-s)x + sy) \leq (1-q)\gamma(x) + q\gamma(y),$$
$$\gamma(rx + (1-r)y) \geq p\gamma(x) + (1-p)\gamma(y).$$

Proof. Similarly as in the proofs of the previous lemmas, assuming that γ is increasing, calculating the mean L from equality (1), and taking into account that $\min(x, y) \leq L(x, y) \leq \max(x, y)$, we obtain inequalities (2) for all $x, y \in I$. Since $r + s > 1$ and $p + q < 1$, we get

$$\gamma^{-1}\left(\frac{\gamma((1-r-s)\max(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right)$$
$$\leq K(x,y) \leq \gamma^{-1}\left(\frac{\gamma((1-r-s)\min(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right)$$

for all $x, y \in I$.

Since K is a mean, it follows that

$$\gamma^{-1}\left(\frac{\gamma((1-r-s)\max(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right) \leq \max(x,y)$$

and

$$\gamma^{-1}\left(\frac{\gamma((1-r-s)\min(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right) \geq \min(x,y),$$

for all $x, y \in I$. Hence, as γ is increasing and $1 - p - q > 0$, we get

$$\gamma((1-r-s)\max(x,y) + rx + sy) \leq (1-p-q)\gamma(\max(x,y)) + p\gamma(x) + q\gamma(y),$$

$x, y \in I$, and

$$\gamma((1-r-s)\min(x,y) + rx + sy) \geq (1-p-q)\gamma(\min(x,y)) + p\gamma(x) + q\gamma(y),$$

$x, y \in I$. Now, if $x < y$ from these inequalities we get

$$\gamma(rx + (1-r)y) \le p\gamma(x) + (1-p)\gamma(y),$$
$$\gamma((1-s)x + sy) \ge (1-q)\gamma(x) + q\gamma(y);$$

and, if $x > y$ we get

$$\gamma((1-s)x + sy) \le (1-q)\gamma(x) + q\gamma(y),$$
$$\gamma(rx + (1-r)y) \ge p\gamma(x) + (1-p)\gamma(y).$$

□

Lemma 4.4. *Let $I \subset \mathbb{R}$ be an interval and $p, q, r, s \in [0, 1]$ be such that $p+q > 1$ and $r + s < 1$. Suppose that $\gamma \in \mathcal{CM}_+(I)$, the functions $K, L : I^2 \to I$ are means in I, and*

$$L_{id}^{(r,s)} = K_\gamma^{(p,q)}.$$

Under these assumptions, for all $x, y \in I$,

(i) *if $x \le y$ then*

$$\gamma((1-s)x + sy) \le p\gamma(x) + (1-p)\gamma(y)$$
$$\gamma(rx + (1-r)y) \ge (1-q)\gamma(x) + q\gamma(y);$$

(ii) *if $x > y$ then*

$$\gamma(rx + (1-r)y) \ge (1-q)\gamma(x) + q\gamma(y),$$
$$\gamma((1-s)x + sy) \le p\gamma(x) + (1-p)\gamma(y).$$

Proof. Assume that γ is increasing. Calculating the mean L from the assumed equality, we obtain inequalities (2) for all $x, y \in I$. Since $r + s < 1$ and $p + q > 1$, we get

$$\gamma^{-1}\left(\frac{\gamma((1-r-s)\min(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right)$$
$$\le K(x,y) \le \gamma^{-1}\left(\frac{\gamma((1-r-s)\max(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right)$$

for all $x, y \in I$. Since K is a mean, it follows that

$$\gamma^{-1}\left(\frac{\gamma((1-r-s)\min(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right) \le \max(x,y)$$

and

$$\gamma^{-1}\left(\frac{\gamma((1-r-s)\max(x,y) + rx + sy) - p\gamma(x) - q\gamma(y)}{1-p-q}\right) \ge \min(x,y),$$

for all $x, y \in I$. Hence, as γ is increasing and $1 - p - q < 0$, we get

$$\gamma((1-r-s)\max(x,y) + rx + sy) \ge (1-p-q)\gamma(\max(x,y)) + p\gamma(x) + q\gamma(y),$$

$x, y \in I$, and

$\gamma((1-r-s)\min(x,y)+rx+sy) \le (1-p-q)\gamma(\min(x,y))+p\gamma(x)+q\gamma(y),$

$x, y \in I$. Now, if $x < y$ from these inequalities we get

$$\begin{aligned}\gamma(rx+(1-r)y) &\ge p\gamma(x)+(1-p)\gamma(y),\\ \gamma((1-s)x+sy) &\le (1-q)\gamma(x)+q\gamma(y);\end{aligned}$$

and, if $x > y$ we get

$$\begin{aligned}\gamma((1-s)x+sy) &\ge (1-q)\gamma(x)+q\gamma(y),\\ \gamma(rx+(1-r)y) &\le p\gamma(x)+(1-p)\gamma(y).\end{aligned}$$

□

5. Auxiliary results on half-convex and half-concave functions

Lemma 5.1. *Let $I \subset \mathbb{R}$ be an interval and let $a, b \in (0,1)$.*

(i) *If $f : I \to \mathbb{R}$ satisfies the inequality*

$$f(ax+(1-a)y) \le bf(x)+(1-b)f(y), \quad x, y \in I,\ x \le y, \tag{3}$$

then

$$f\left(\sum_{j=0}^{n}\binom{n}{j}a^{n-j}(1-a)^j x_j\right) \le \sum_{j=0}^{n}\binom{n}{j}b^{n-j}(1-b)^j f(x_j) \tag{4}$$

for all $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n \in I x_0 \le x_1 \le \cdots \le x_n$.

Moreover, if (3) is satisfied for all $x, y \in I$, then (4) holds for all $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n \in I$.

(ii) *If $f : I \to \mathbb{R}$ satisfies the inequality*

$$f(ax+(1-a)y) \le bf(x)+(1-b)f(y), \quad x, y \in I,\ x \ge y, \tag{5}$$

then

$$f\left(\sum_{j=0}^{n}\binom{n}{j}a^{n-j}(1-a)^j x_j\right) \le \sum_{j=0}^{n}\binom{n}{j}b^{n-j}(1-b)^j f(x_j) \tag{6}$$

for all $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n \in I x_0 \ge x_1 \ge \cdots \ge x_n$.

Moreover, if (5) is satisfied for all $x, y \in I$, then (6) holds for all $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n \in I$.

Proof. It is enough to prove (i). By (3) inequality (4) holds true for $n = 1$. Assume that (4) holds true for some $n \in \mathbb{N}$ and take arbitrary $x_0, x_1, \ldots, x_{n+1} \in I, x_0 \le x_1 \le \cdots \le x_{n+1}$. Applying first inequality (4) with x_j replaced by $ax_j + (1-a)x_{j+1}$ and then inequality (3), we obtain

$$
\begin{aligned}
& f\left(\sum_{j=0}^{n+1}\binom{n+1}{j}a^{n+1-j}(1-a)^j x_j\right) \\
&\quad = f\left(\sum_{j=0}^{n}\binom{n}{j}a^{n-j}(1-a)^j (ax_j+(1-a)x_{j+1})\right) \\
&\quad \le \sum_{j=0}^{n}\binom{n}{j}b^{n-j}(1-b)^j f(ax_j+(1-a)x_{j+1}) \\
&\quad \le \sum_{j=0}^{n}\binom{n}{j}b^{n-j}(1-b)^j [bf(x_j)+(1-b)f(x_{j+1})] \\
&\quad = \sum_{j=0}^{n+1}\binom{n+1}{j}b^{n+1-j}(1-b)^j f(x_j),
\end{aligned}
$$

and induction completes the proof. □

Remark 5.2. Obviously, if inequality (3) (or (5)) is reversed, the counterpart of this lemma remains true.

Lemma 5.3. *Let* $a \in (0,1)$ *be fixed. Then the set*

$$A := \left\{\sum_{j=0}^{k}\binom{n}{j}a^{n-j}(1-a)^j : n \in \mathbb{N};\ k \in \{0,1\ldots,n\}\right\}$$

is dense in $[0,1]$.

Proof. Note that $A = \bigcup_{n\in\mathbb{N}} A_n$ where, for each $n \in \mathbb{N}$,

$$A_n = \left\{\sum_{j=0}^{k}\binom{n}{j}a^{n-j}(1-a)^j :\ k \in \{0,1,\ldots,n\}\right\}$$

forms a finite increasing sequence with the first value being equal to a^n (tending to 0 as $n \to \infty$), the last value 1, and the distance between two successive values:

$$\sum_{j=0}^{k}\binom{n}{j}a^{n-j}(1-a)^j - \sum_{j=0}^{k-1}\binom{n}{j}a^{n-j}(1-a)^j = \binom{n}{k}a^{n-k}(1-a)^k$$

tending to 0 as $n \to \infty$; it easily follows from the fact that, for any $k \in \mathbb{N}$,

$$\lim_{n\to\infty}\frac{\binom{n+1}{k}a^{n+1-k}(1-a)^k}{\binom{n}{k}a^{n-k}(1-a)^k} = a < 1.$$

□

Lemma 5.4. *Let $I \subset \mathbb{R}$ be an interval and $p, s \in (0,1)$ be fixed. Suppose that $\gamma : I \to \mathbb{R}$ is a continuous function.*

(i) *If for all $x, y \in I, x \leq y$,*

$$\gamma((1-s)x + sy) \leq p\gamma(x) + (1-p)\gamma(y), \tag{7}$$

then there exists a function $\lambda : [0,1] \to [0,1]$, such that for all $x, y \in I$, $x \geq y$,

$$\gamma(tx + (1-t)y) \leq \lambda(t)\gamma(x) + (1-\lambda(t))\gamma(y). \tag{8}$$

(ii) *If for all $x, y \in I, x \geq y$,*

$$\gamma((1-s)x + sy) \geq p\gamma(x) + (1-p)\gamma(y), \tag{9}$$

then there exists a function $\lambda : [0,1] \to [0,1]$, such that for all $x, y \in I, x \geq y$,

$$\gamma(tx + (1-t)y) \geq \lambda(t)\gamma(x) + (1-\lambda(t))\gamma(y). \tag{10}$$

(iii) *If inequality (7) is satisfied for all $x, y \in I, x \leq y$, and inequality (9) is satisfied for all $x, y \in I$, $x \geq y$, then there exists a function $\lambda : [0,1] \to [0,1], \lambda([0,1]) = [0,1]$, such that inequality (8) is satisfied for all for $x, y \in I, x \leq y$; and inequality (10) is satisfied for all $x, y \in I, x \geq y$.*

Proof. (i) Setting $a := 1-s$ and $b := p$ we see that the function $f := \gamma$ satisfies inequality (3). By Lemma 5.1(i) we get

$$\gamma\left(\sum_{j=0}^{n}\binom{n}{j}a^{n-j}(1-a)^j x_j\right) \leq \sum_{j=0}^{n}\binom{n}{j}b^{n-j}(1-b)^j\gamma(x_j)$$

for all $n \in \mathbb{N}$ and $x_0, x_1, \ldots, x_n \in I$, $x_0 \leq x_1 \leq \cdots \leq x_n$.

Take arbitrary $x, y \in I, x < y$, and $k, n \in \mathbb{N}, k \leq n$. Setting here

$$x_0 = x_1 = \cdots = x_k := x; \quad x_{k+1} = x_{k+2} = \cdots = x_n := y$$

we obtain

$$\gamma(a_{n,k}x + (1-a_{n,k})y) \leq b_{n,k}\gamma(x) + (1-b_{n,k})\gamma(y),$$

where

$$a_{n,k} := \left(\sum_{j=0}^{k}\binom{n}{j}a^{n-j}(1-a)^j\right), \quad b_{n,k} := \left(\sum_{j=0}^{k}\binom{n}{j}b^{n-j}(1-b)^j\right).$$

By Lemma 5.3, the sets $A := \{a_{n,k} : n \in \mathbb{N},\ k = 0, \ldots, n\}$ and $B := \{b_{n,k} : n \in \mathbb{N}, k = 0, \ldots, n\}$ are dense in $(0,1)$. Hence, for arbitrary $t \in (0,1)$, there is a sequence $t_j \in A, t_j = a_{n_j,k_j}, j \in \mathbb{N}$, such that $\lim_{j\to\infty} t_j = t$ and

$$\gamma(t_j x + (1-t_j)y) \leq b_{n_j,k_j}\gamma(x) + (1-b_{n_j,k_j})\gamma(y).$$

Setting a convergent subsequence (b_{j_m}) from the sequence $(b_{n_j,k_j})_j$ and setting here $\lambda(t) := \lim_{m\to\infty} b_{j_m}$, we obtain

$$\gamma(tx + (1-t)y) \le \lambda(t)\gamma(x) + (1-\lambda(t))\gamma(y),$$

which was to be shown.

Note that in case (ii), with the same notations and the definition of the function λ, applying Lemma 5.1(ii), in a similar way, we first obtain the inequality

$$\gamma(a_{n,k}x + (1-a_{n,k})y) \ge b_{n,k}\gamma(x) + (1-b_{n,k})\gamma(y)$$

for all $x, y \in I, x < y$, and $k, n \in \mathbb{N}, k \le n$, and then

$$\gamma(tx + (1-t)y) \ge \lambda(t)\gamma(x) + (1-\lambda(t))\gamma(y)$$

for $t \in (0,1)$.

Part (iii) follows from parts (i) and (ii). $\square$

In the next basic result we use parts (i) and (iii) of Lemma 5.4.

Lemma 5.5. *Let $I \subset \mathbb{R}$ be an interval. If a function $\gamma \in \mathcal{CM}_+(I)$ is such that for every $t \in (0,1)$ there exists $\lambda(t) \in (0,1)$ such that for all $x, y \in I, x \le y$,*

$$\gamma(tx + (1-t)y) \le \lambda(t)\gamma(x) + (1-\lambda(t))\gamma(y);$$

and for all $x, y \in I, x \ge y$,

$$\gamma(tx + (1-t)y) \ge \lambda(t)\gamma(x) + (1-\lambda(t))\gamma(y),$$

then there are $a, b \in \mathbb{R}, a \ne 0$, such that

$$\gamma(x) = ax + b, \quad x \in I.$$

Proof. Take $t = \frac{1}{2}$ and arbitrary $x, y \in I, x < y$. In view of the first assumed inequality we have

$$\gamma\left(\frac{x+y}{2}\right) \le \lambda\left(\frac{1}{2}\right)\gamma(x) + \left(1-\lambda\left(\frac{1}{2}\right)\right)\gamma(y), \tag{11}$$

and from the second inequality (interchanging the roles of x and y)

$$\gamma\left(\frac{y+x}{2}\right) \ge \lambda\left(\frac{1}{2}\right)\gamma(y) + \left(1-\lambda\left(\frac{1}{2}\right)\right)\gamma(x). \tag{12}$$

It follows that

$$\lambda\left(\frac{1}{2}\right)\gamma(y) + \left(1-\lambda\left(\frac{1}{2}\right)\right)\gamma(x) \le \lambda\left(\frac{1}{2}\right)\gamma(x) + \left(1-\lambda\left(\frac{1}{2}\right)\right)\gamma(y),$$

whence

$$\left(2\lambda\left(\frac{1}{2}\right) - 1\right)(\gamma(y) - \gamma(x)) \le 0.$$

Since γ is increasing, we conclude that

$$\lambda\left(\frac{1}{2}\right) \leq \frac{1}{2}.$$

Similarly, for arbitrary $x, y \in I, x > y$, from the second assumed inequality we have

$$\gamma\left(\frac{x+y}{2}\right) \geq \lambda\left(\frac{1}{2}\right)\gamma(x) + \left(1 - \lambda\left(\frac{1}{2}\right)\right)\gamma(y), \tag{13}$$

and from the first inequality (where x and y are interchanged) we get

$$2\gamma\left(\frac{y+x}{2}\right) \leq \lambda\left(\frac{1}{2}\right)\gamma(y) + \left(1 - \lambda\left(\frac{1}{2}\right)\right)\gamma(x). \tag{14}$$

whence

$$\lambda\left(\frac{1}{2}\right)\gamma\left(x\right) + \left(1 - \lambda\left(\frac{1}{2}\right)\right)\gamma\left(y\right) \leq \lambda\left(\frac{1}{2}\right)\gamma\left(y\right) + \left(1 - \lambda\left(\frac{1}{2}\right)\right)\gamma\left(x\right),$$

which can be written in the form

$$\left(2\lambda\left(\frac{1}{2}\right) - 1\right)\left(\gamma\left(x\right) - \gamma\left(y\right)\right) \leq 0.$$

Since γ is strictly increasing we obtain that

$$\lambda\left(\frac{1}{2}\right) \geq \frac{1}{2},$$

and consequently

$$\lambda\left(\frac{1}{2}\right) = \frac{1}{2}.$$

Now inequalities (11), (12), (13) and (14) imply that

$$\gamma\left(\frac{x+y}{2}\right) = \frac{\gamma\left(x\right) + \gamma\left(y\right)}{2}, \quad x, y \in I,$$

that is γ is Jensen affine. The strict monotonicity of γ implies that $\gamma(x) = ax + b (x \in I)$, for some $a, b \in \mathbb{R}, a \neq 0$. This completes the proof. □

Remark 5.6. Clearly, the counterparts of this lemma in all the remaining cases remain true.

6. Main results

Theorem 6.1. *Let $I \subset \mathbb{R}$ be an interval and $p, q, r, s \in [0, 1]$ such that $p + q \neq 1 \neq r + s$. Suppose that $\gamma \in \mathcal{CM}(I)$ and the functions $K, L : I^2 \to I$ are means in I.*

Then $L_{id}^{(r,s)} = K_{\gamma}^{(p,q)}$ *if, and only if,* $\gamma \sim id$ *and*

$$L(x,y) = \frac{(p-r)x + (q-s)y + (1-p-q)K(x,y)}{1-r-s}, \quad x,y \in I. \tag{15}$$

Proof. Without any loss of generality we can assume that γ is increasing. Assume that $L_{id}^{(r,s)} = K_{\gamma}^{(p,q)}$. Of course the numbers $p,q,r,s \in [0,1]$ such that $p+q \neq 1 \neq r+s$ must satisfy the assumptions of exactly one of Lemmas 4.1–4.4. Applying this lemma together with Lemmas 5.4 and 5.5, we conclude that there are $a,b \in \mathbb{R}, a \neq 0$, such that

$$\gamma(x) = ax + b, \quad x \in I,$$

that is $\gamma \sim id$. Setting this function γ into equality $L_{id}^{(r,s)} = K_{\gamma}^{(p,q)}$ we obtain

$$rx + sy + (1-r-s)L(x,y) = px + qy + (1-p-q)K(x,y), \quad x,y \in I,$$

which implies (15). Since the converse implication is obvious, the proof is complete. □

Hence, in view of Remark 3.2, we obtain

Theorem 6.2. *Let* $I \subset \mathbb{R}$ *be an interval,* $K, L : I^2 \to I$ *be means in* $I, p,q,r,s \in [0,1]$ *be such that* $p+q \neq 1 \neq r+s$*, and* $\varphi, \psi \in \mathcal{CM}(I)$*. Then*

$$L_{\psi}^{(r,s)} = K_{\varphi}^{(p,q)}$$

if, and only if, $\psi \sim \varphi$ *and*

$$L(x,y) = \varphi^{-1}\left(\frac{(p-r)\varphi(x) + (q-s)\varphi(y) + (1-p-q)\varphi(K(x,y))}{1-r-s}\right)$$

for all $x,y \in I$.

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